

MOMENTS AND ENTROPY OF THE INTERPOLATING FAMILY OF SIZE DISTRIBUTIONS

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ABSTRACT. Sinner et al. (2016) recently introduced a five-parameter family of size distributions, coined *Interpolating Family* or *IF distribution* for short. In this complementary note, we take advantage of the tractability of the IF distribution to compute the moments and the differential entropy. As a consequence, we deduce at a single stroke the corresponding expressions for many well-known size distributions arising as special cases of the IF distribution.

Keywords: Interpolating Family, Moments, Differential Entropy.

1. INTRODUCTION

Sinner et al. (2016) recently introduced a five-parameter family of size distributions, called *Interpolating Family* or *IF distribution*. In this note, we give explicit expressions for the moments and the differential entropy of its three main subfamilies, denoted IF1, IF2 and IF3 distributions.

The probability density function (pdf) of the IF distribution is given by

$$f_{p,b}(x) = \text{sign}(b)q \, g(x)G(x)^{-q-1} \left(1 - \frac{1}{p+1}G(x)^{-q}\right)^p,$$

where $G(x) = (p+1)^{-\frac{1}{q}} + \left(\frac{x-x_0}{c}\right)^b$, $g(x) = \frac{d}{dx}G(x) = \frac{b}{c}\left(\frac{x-x_0}{c}\right)^{b-1}$ and $x \geq x_0$. The distribution depends on five parameters: an interpolation parameter $p \in [0, \infty]$, a shape parameter $b \in \mathbb{R}_0$, a scale parameter $c > 0$, a tail-weight parameter $q > 0$ and a location parameter $x_0 \geq 0$. Three interesting four-parameter subfamilies are nested inside this five-parameter family. If $p = 0$, we get a power law distribution, called *Interpolating Family of the first kind (IF1)* with pdf

$$f_{p,b}(x) = \frac{|b|q}{c} \left(\frac{x-x_0}{c}\right)^{b-1} \left(1 + \left(\frac{x-x_0}{c}\right)^b\right)^{-q-1},$$

where $x \geq x_0$. The IF1 distribution contains many well-known distributions such as for example the Pareto type I, II, III and IV, Lomax, Fisk and Burr type III and XII distributions (Burr, 1942; Tadikamalla, 1980; Lindsay et al., 1996). If, on the other hand, $p \rightarrow \infty$, we get a power law distribution with exponential cut-off, called *Interpolating Family of the second kind (IF2)* with pdf

$$f_{\infty,b}(x) = \frac{|b|q}{c} \left(\frac{x-x_0}{c}\right)^{-bq-1} e^{-\left(\frac{x-x_0}{c}\right)^{-bq}},$$

where $x \geq x_0$. Special cases of the IF2 distribution are the Weibull, Fréchet, Gumbel type II, Rayleigh and Exponential distributions. Finally, if $b = 1$, we get

the *Interpolating Family of the third kind (IF3)* with pdf

$$f_{p,1}(x) = \frac{q}{c} \left((p+1)^{-\frac{1}{q}} + \frac{x-x_0}{c} \right)^{-q-1} \left(1 - \left(1 + (p+1)^{\frac{1}{q}} \frac{x-x_0}{c} \right)^{-q} \right)^p,$$

where $x \geq x_0$. The IF3 distribution contains most notably the Generalized Lomax and the Stoppa distributions (Kleiber and Kotz, 2003).

2. MOMENTS

The r^{th} moment of the IF distribution is given by

$$\mathbb{E}[X^r] = \int_{x_0}^{\infty} x^r f_{p,b}(x) dx.$$

If we make the change of variables $y = (p+1)^{-\frac{1}{q}} + \left(\frac{x-x_0}{c}\right)^b$ and apply Newton's binomial theorem, then we get

$$\mathbb{E}[X^r] = \sum_{i=0}^r \binom{r}{i} x_0^i c^{r-i} \underbrace{\int_{(p+1)^{-\frac{1}{q}}}^{\infty} q y^{-q-1} \left(y - (p+1)^{-\frac{1}{q}}\right)^{\frac{r-i}{b}} \left(1 - \frac{y^{-q}}{p+1}\right)^p dy}_{I(p,b,q)}.$$

We compute the integral $I(p, b, q)$ for the three subfamilies IF1, IF2 and IF3.

IF1 distribution. If we plug in $p = 0$ and set $z = \frac{1}{y}$, then we get

$$I(0, b, q) = \int_1^{\infty} q y^{-q-1} (y-1)^{\frac{r-i}{b}} dy = q \int_0^1 z^{q-1-\frac{r-i}{b}} (1-z)^{\frac{r-i}{b}} dz.$$

If either $b > 0$ and $r < bq$ or else $b < 0$ and $r < -b$, then this can be written as

$$I(0, b, q) = qB\left(q - \frac{r-i}{b}, 1 + \frac{r-i}{b}\right) = \frac{\Gamma\left(q - \frac{r-i}{b}\right)\Gamma\left(1 + \frac{r-i}{b}\right)}{\Gamma(q)}.$$

The r^{th} moment of the IF1 distribution is given by

$$\mathbb{E}[X^r] = \sum_{i=0}^r \binom{r}{i} x_0^i c^{r-i} \frac{\Gamma\left(q - \frac{r-i}{b}\right)\Gamma\left(1 + \frac{r-i}{b}\right)}{\Gamma(q)} \quad \text{if } \begin{cases} b > 0 \text{ and } r < bq, \\ b < 0 \text{ and } r < -b. \end{cases}$$

IF2 distribution. If we make the change of variables $z = y^{-q}$, then we get

$$\lim_{p \rightarrow \infty} I(p, b, q) = \lim_{p \rightarrow \infty} \int_0^{p+1} \left(z^{-\frac{1}{q}} - (p+1)^{-\frac{1}{q}}\right)^{\frac{r-i}{b}} \left(1 - \frac{z}{p+1}\right)^p dz.$$

We may apply the Lebesgue dominated convergence theorem to deduce that

$$\lim_{p \rightarrow \infty} I(p, b, q) = \int_0^{\infty} z^{-\frac{r-i}{bq}} e^{-z} dz = \Gamma\left(1 - \frac{r-i}{bq}\right).$$

The r^{th} moment of the IF2 distribution is given by

$$\mathbb{E}[X^r] = \sum_{i=0}^r \binom{r}{i} x_0^i c^{r-i} \Gamma\left(1 - \frac{r-i}{bq}\right) \quad \text{if } \begin{cases} b > 0 \text{ and } r < bq, \\ b < 0. \end{cases}$$

IF3 distribution. If $b = 1$, then we make the change of variables $z = \frac{y^{-q}}{p+1}$. Using Newton's binomial theorem, we get

$$\begin{aligned} I(p, 1, q) &= (p+1)^{1-\frac{r-i}{q}} \int_0^1 \left(z^{-\frac{1}{q}} - 1\right)^{r-i} (1-z)^p dz \\ &= (p+1)^{1-\frac{r-i}{q}} \sum_{k=0}^{r-i} \binom{r-i}{k} (-1)^k \int_0^1 z^{-\frac{1}{q}(r-i-k)} (1-z)^p dz. \end{aligned}$$

If $r < q$, then this can be written as

$$I(p, 1, q) = (p+1)^{1-\frac{r-i}{q}} \sum_{k=0}^{r-i} \binom{r-i}{k} (-1)^k B\left(1 - \frac{1}{q}(r-i-k), p+1\right).$$

Under the hypothesis $r < q$, the r^{th} moment of the IF3 distribution is given by

$$\mathbb{E}[X^r] = \sum_{i=0}^r \binom{r}{i} x_0^i c^{r-i} (p+1)^{1-\frac{r-i}{q}} \sum_{k=0}^{r-i} \binom{r-i}{k} (-1)^k B\left(1 - \frac{1}{q}(r-i-k), p+1\right).$$

3. DIFFERENTIAL ENTROPY

The differential entropy h of a continuous random variable with pdf f is

$$h(f) = - \int_S f(x) \ln(f(x)) dx, \quad (1)$$

where S is the support of the random variable (Shannon, 1948). We will show that the differential entropy of the IF distribution is given by

$$h(f_{p,b}) = -\ln\left(\frac{|b|q}{c}\right) - \frac{b-1}{b} F(p, q) - \frac{bq+1}{bq} \ln(p+1) + \frac{q+1}{q} H_{p+1} + \frac{p}{p+1},$$

where $H_{p+1} = \sum_{k=1}^{p+1} \frac{1}{k}$ is the $(p+1)^{\text{th}}$ harmonic number and

$$F(p, q) = (p+1) \int_0^1 \ln\left(t^{-\frac{1}{q}} - 1\right) (1-t)^p dt.$$

To prove this result, we proceed as follows. If $0 \leq p < \infty$, then we apply the change of variables $t = \left(1 + (p+1)^{\frac{1}{q}} \left(\frac{x-x_0}{c}\right)^b\right)^{-q}$ to (1):

$$\begin{aligned} h(f_{p,b}) &= -\ln\left(\frac{|b|q}{c}\right) - \frac{b-1}{b} (p+1) \int_0^1 \ln(t^{-\frac{1}{q}} - 1) (1-t)^p dt \\ &\quad - \frac{bq+1}{bq} \ln(p+1) (p+1) \int_0^1 (1-t)^p dt \\ &\quad - \frac{q+1}{q} (p+1) \int_0^1 \ln(t) (1-t)^p dt - p(p+1) \int_0^1 \ln(1-t) (1-t)^p dt. \end{aligned}$$

Clearly, $(p+1) \int_0^1 (1-t)^p dt = 1$ and $(p+1) \int_0^1 \ln(1-t) (1-t)^p dt = -\frac{1}{p+1}$. Moreover,

integrating by parts, we deduce that $(p+1) \int_0^1 \ln(t) (1-t)^p dt = -H_{p+1}$ and the result follows. In particular, the differential entropy of the IF1 distribution ($p = 0$) is given by

$$h(f_{0,b}) = -\ln\left(\frac{|b|q}{c}\right) + \frac{b-1}{b} H_{q-1} + \frac{q+1}{q}$$

and the differential entropy of the IF3 distribution ($0 < p < \infty$ and $b = 1$) is

$$h(f_{p,1}) = -\ln\left(\frac{q}{c}\right) + \frac{q+1}{q} (H_{p+1} - \ln(p+1)) + \frac{p}{p+1}.$$

On the other hand, if $p \rightarrow \infty$, then we make the change of variables $z = \left(\frac{x-x_0}{c}\right)^{-bq}$ in (1) and get

$$h(f_{\infty,b}) = -\ln\left(\frac{|b|q}{c}\right) - \frac{bq+1}{bq} \int_0^\infty e^{-z} \ln(z) \, dz + \int_0^\infty z e^{-z} \, dz.$$

We can express the integral in the second term as the Euler–Mascheroni constant

$$\int_0^\infty e^{-z} \ln(z) \, dz = -\gamma_E$$

and the integral in the third term can be simplified to

$$\int_0^\infty z e^{-z} \, dz = \Gamma(2) = 1.$$

We conclude that the differential entropy of the IF2 distribution ($p \rightarrow \infty$) becomes

$$h(f_{\infty,b}) = -\ln\left(\frac{|b|q}{c}\right) + \frac{bq+1}{bq} \gamma_E + 1,$$

where $\gamma_E = \lim_{p \rightarrow \infty} (H_{p+1} - \ln(p+1))$.

By convexity (Michalowicz et al., 2014), we deduce that the IF subfamilies maximize the differential entropy within the class of all continuous probability distributions under the following constraints:

Corollary 1. *The IF1 distribution maximizes the differential entropy within the class of all continuous probability distributions satisfying the constraints*

$$\begin{aligned} \mathbb{E} \left[\ln\left(\frac{x-x_0}{c}\right) \right] &= -\frac{H_q-1}{b} \\ \mathbb{E} \left[\ln\left(1 + \left(\frac{x-x_0}{c}\right)^b\right) \right] &= \frac{1}{q} \end{aligned}$$

and having support $x \geq x_0$.

Corollary 2. *The IF2 distribution maximizes the differential entropy within the class of all continuous probability distributions satisfying the constraints*

$$\begin{aligned} \mathbb{E} \left[\ln\left(\frac{x-x_0}{c}\right) \right] &= \frac{\gamma_E}{bq} \\ \mathbb{E} \left[\left(\frac{x-x_0}{c}\right)^{-bq} \right] &= 1 \end{aligned}$$

and having support $x \geq x_0$.

Corollary 3. *The IF3 distribution maximizes the differential entropy within the class of all continuous probability distributions satisfying the constraints*

$$\begin{aligned} \mathbb{E} \left[\ln \left((p+1)^{-\frac{1}{q}} + \frac{x-x_0}{c} \right) \right] &= \frac{1}{q} (H_{p+1} - \ln(p+1)) \\ \mathbb{E} \left[\ln \left(1 - \left(1 + (p+1)^{\frac{1}{q}} \frac{x-x_0}{c} \right)^{-q} \right) \right] &= -\frac{1}{p+1} \end{aligned}$$

and having support $x \geq x_0$.

4. MEAN AND ENTROPY OF SPECIAL CASES

In the two following Tables, we display the mean and the differential entropy for some of the size distributions arising as special cases of the Interpolating Family. While most of these already appeared somewhere in the literature (see for example Michalowicz et al. (2014) and Yari and Mohtashami Borzadaran (2010)), we find it instructive to assemble them as done in the following Tables.

Distribution name	# par.	Parameters (p, b, c, q, x_0)	Mean $\mathbb{E}[X]$	Constraint
Pareto IV	4	$(0, \frac{1}{\gamma} > 0, c, q, x_0)$	$x_0 + c \frac{\Gamma(q-\gamma)\Gamma(1+\gamma)}{\Gamma(q)}$	$q > \gamma$
Lindsay–Burr III	4	$(0, b < 0, c, q, x_0)$	$x_0 + c \frac{\Gamma(q-\frac{1}{b})\Gamma(1+\frac{1}{b})}{\Gamma(q)}$	$b < -1$
Pareto II	3	$(0, 1, c, q, x_0)$	$x_0 + \frac{c}{q-1}$	$q > 1$
Pareto III	3	$(0, \frac{1}{\gamma} > 0, c, 1, x_0)$	$x_0 + c\Gamma(1-\gamma)\Gamma(1+\gamma)$	$\gamma < 1$
Tadikamalla–Burr XII	3	$(0, b > 0, c, q, 0)$	$c \frac{\Gamma(q-\frac{1}{b})\Gamma(1+\frac{1}{b})}{\Gamma(q)}$	$bq > 1$
Fisk	2	$(0, b > 0, c, 1, 0)$	$c\Gamma(1-\frac{1}{b})\Gamma(1+\frac{1}{b})$	$b > 1$
Lomax	2	$(0, 1, c, q, 0)$	$\frac{c}{q-1}$	$q > 1$
Pareto I	2	$(0, 1, x_0, q, x_0)$	$\frac{q}{q-1}x_0$	$q > 1$
Burr XII	2	$(0, b > 0, 1, q, 0)$	$\frac{\Gamma(q-\frac{1}{b})\Gamma(1+\frac{1}{b})}{\Gamma(q)}$	$bq > 1$
Weibull	3	$(\infty, -1, c, q, x_0)$	$x_0 + c\Gamma\left(1 + \frac{1}{q}\right)$	
Fréchet	3	$(\infty, 1, c, q, x_0)$	$x_0 + c\Gamma\left(1 - \frac{1}{q}\right)$	$q > 1$
Gumbel II	2	$(\infty, 1, c, q, 0)$	$c\Gamma\left(1 - \frac{1}{q}\right)$	$q > 1$
Rayleigh	1	$(\infty, -1, c, 2, 0)$	$\frac{c}{\sqrt{2}}\sqrt{\frac{\pi}{2}}$	
Exponential	1	$(\infty, -1, c, 1, 0)$	c	
Generalized Lomax	3	$(m-1, 1, c, q, 0)$	$c m^{1-\frac{1}{q}} \left(B\left(1 - \frac{1}{q}, m\right) - B\left(1 + \frac{1}{q}, m\right) \right)$	$q > 1$
Stoppa	3	$(m-1, 1, c, q, cm^{-\frac{1}{q}})$	$x_0 m B\left(1 - \frac{1}{q}, m\right)$	$q > 1$

Distribution name	Parameters (p, b, c, q, x_0)	Entropy $h(f)$	Maximum entropy constraints	
Pareto IV	$(0, \frac{1}{\gamma} > 0, c, q, x_0)$	$(1 - \gamma)H_{q-1} + \frac{q+1}{q} - \ln\left(\frac{q}{c\gamma}\right)$	$\mathbb{E}[\ln(\frac{x-x_0}{c})] = -\gamma H_{q-1}$	$\mathbb{E}[\ln(1 + (\frac{x-x_0}{c})^{\frac{1}{\gamma}})] = \frac{1}{q}$
Lindsay–Burr III	$(0, b < 0, c, q, x_0)$	$\frac{b-1}{b}H_{q-1} + \frac{q+1}{q} - \ln\left(\frac{ b q}{c}\right)$	$\mathbb{E}[\ln(\frac{x-x_0}{c})] = -\frac{H_{q-1}}{b}$	$\mathbb{E}[\ln(1 + (\frac{x-x_0}{c})^b)] = \frac{1}{q}$
Pareto II	$(0, 1, c, q, x_0)$	$\frac{q+1}{q} - \ln\left(\frac{q}{c}\right)$	$\mathbb{E}[\ln(\frac{x-x_0}{c})] = -H_{q-1}$	$\mathbb{E}[\ln(1 + \frac{x-x_0}{c})] = \frac{1}{q}$
Pareto III	$(0, \frac{1}{\gamma} > 0, c, 1, x_0)$	$2 + \ln(c\gamma)$	$\mathbb{E}[\ln(\frac{x-x_0}{c})] = 0$	$\mathbb{E}[\ln(1 + (\frac{x-x_0}{c})^{\frac{1}{\gamma}})] = 1$
Tadikamalla–Burr XII	$(0, b > 0, c, q, 0)$	$\frac{b-1}{b}H_{q-1} + \frac{q+1}{q} - \ln(\frac{ b q}{c})$	$\mathbb{E}[\ln(\frac{x}{c})] = -\frac{H_{q-1}}{b}$	$\mathbb{E}[\ln(1 + (\frac{x}{c})^b)] = \frac{1}{q}$
Fisk	$(0, b > 0, c, 1, 0)$	$2 - \ln(\frac{b}{c})$	$\mathbb{E}[\ln(\frac{x}{c})] = 0$	$\mathbb{E}[\ln(1 + (\frac{x}{c})^b)] = 1$
Lomax	$(0, 1, c, q, 0)$	$\frac{q+1}{q} - \ln(\frac{q}{c})$	$\mathbb{E}[\ln(\frac{x}{c})] = -H_{q-1}$	$\mathbb{E}[\ln(1 + \frac{x}{c})] = \frac{1}{q}$
Pareto I	$(0, 1, x_0, q, x_0)$	$\frac{q+1}{q} - \ln(\frac{q}{x_0})$	$\mathbb{E}[\ln(\frac{x}{c} - 1)] = -H_{q-1}$	$\mathbb{E}[\ln(\frac{x}{x_0})] = \frac{1}{q}$
Burr XII	$(0, b > 0, 1, q, 0)$	$\frac{b-1}{b}H_{q-1} + \frac{q+1}{q} - \ln(bq)$	$\mathbb{E}[\ln(x)] = -\frac{H_{q-1}}{b}$	$\mathbb{E}[\ln(1 + x^b)] = \frac{1}{q}$
Weibull	$(\infty, -1, c, q, x_0)$	$\frac{q-1}{q}\gamma_E + 1 - \ln(\frac{q}{c})$	$\mathbb{E}[\ln(\frac{x-x_0}{c})] = -\frac{\gamma_E}{q}$	$\mathbb{E}[(x - x_0)^q] = c^q$
Fréchet	$(\infty, 1, c, q, x_0)$	$\frac{q+1}{q}\gamma_E + 1 - \ln(\frac{q}{c})$	$\mathbb{E}[\ln(\frac{x-x_0}{c})] = \frac{\gamma_E}{q}$	$\mathbb{E}[(x - x_0)^{-q}] = c^{-q}$
Gumbel II	$(\infty, 1, c, q, 0)$	$\frac{q+1}{q}\gamma_E + 1 - \ln(\frac{q}{c})$	$\mathbb{E}[\ln(\frac{x}{c})] = \frac{\gamma_E}{q}$	$\mathbb{E}[x^{-q}] = c^{-q}$
Rayleigh	$(\infty, -1, c, 2, 0)$	$\frac{1}{2}\gamma_E + \ln(\frac{c}{2}) + 1$	$\mathbb{E}[\ln(\frac{x}{c})] = -\frac{\gamma_E}{2}$	$\mathbb{E}[x^2] = c^2$
Exponential	$(\infty, -1, c, 1, 0)$	$\ln(c) + 1$		$\mathbb{E}[x] = c$
Generalized Lomax	$(m - 1, 1, c, q, 0)$	$\frac{q+1}{q}(H_m - \ln(m)) + \frac{m-1}{m} - \ln(\frac{q}{c})$	$\mathbb{E}[\ln(m^{-\frac{1}{q}} + \frac{x}{c})] = \frac{1}{q}(H_m - \ln(m))$	$\mathbb{E}[\ln(1 - (1 + m^{\frac{1}{q}}\frac{x}{c})^{-q})] = -\frac{1}{m}$
Stoppa	$(m - 1, 1, c, q, cm^{-\frac{1}{q}})$	$\frac{q+1}{q}(H_m - \ln(m)) + \frac{m-1}{m} - \ln(\frac{q}{c})$	$\mathbb{E}[\ln(\frac{x}{c})] = \frac{1}{q}(H_m - \ln(m))$	$\mathbb{E}[\ln(1 - (m^{-1}\frac{x}{c})^{-q})] = -\frac{1}{m}$

5. CONCLUSION

The first aim of this complementary note was to provide closed form expressions for the moments of the IF1, IF2 and IF3 distributions introduced in Sinner et al. (2016). The second aim was to calculate the differential entropy of the IF distribution and of its subfamilies.

REFERENCES

- Burr, I. W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13:215–232.
- Kleiber, C. and Kotz, S. (2003). *Statistical size distributions in economics and actuarial sciences*. Wiley Series in Probability and Statistics.
- Lindsay, S., Wood, G., and Woollons, R. (1996). Modelling the diameter distribution of forest stands using the Burr distribution. *Journal of Applied Statistics*, 23(6):609–620.
- Michalowicz, J. V., Nichols, J. M., and Bucholtz, F. (2014). *Handbook of differential entropy*. CRC Press, Boca Raton, FL.
- Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, 27:379–423, 623–656.
- Sinner, C., Dominicy, Y., Ley, C., Trufin, J., and Weber, P. (2016). An Interpolating Family of Size Distributions. *arxiv:1606.04430*.
- Tadikamalla, P. R. (1980). A look at the Burr and related distributions. *International Statistical Review. Revue Internationale de Statistique*, 48(3):337–344.
- Yari, G.-H. and Mohtashami Borzadaran, G. R. (2010). Entropy for Pareto-types and its order statistics distributions. *Communications in Information and Systems*, 10(3):193–201.